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A virial theorem for general relativistic charged fluids

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Abstract. We formulate an exact form of the virial theorem for a relativistic charged thermodynamic perfect fluid in curved spacetime in time-orthogonal coordinates and diagonal metric tensor. Its Newtonian limit leads to a generalisation of Chandrasekhar's tensor virial theorem in hydromagnetics. We apply the exact form of the virial theorem in curved spacetime, to obtain equilibrium configurations in two cases.

1. Introduction

Chandrasekhar obtained a tensor form of the virial theorem in Newtonian hydromagnetics (Chandrasekhar 1960, 1961) which he later extended to the case of hydrodynamics in the post-Newtonian approximation of general relativity (Chandrasekhar 1965). The formulation of an exact form of the virial theorem in curved spacetime which will be sufficiently general to include the case of a perfect fluid with or without electromagnetic fields is desirable mainly because of its relevance to relativistic astrophysics.

Detailed comparisons with earlier attempts to extend the virial theorem to the general relativistic regime (Bonazzola 1973, Vilain 1979) are somewhat difficult to make and not very useful, because the point of view adopted in these attempts is different from ours. The scopes of these papers are also different. The virial theorems obtained are scalar and suitable for axisymmetric and spherically symmetric spacetimes. In this paper, scalar and non-scalar forms of the virial theorem are obtained, and no particular spatial symmetries are assumed. A major difference is that the virial theorems of these earlier attempts arise from the integration of the field equations, but the virial theorem obtained here arises from the integration of the products of the equations of motion and the x^i. Furthermore, we explicitly include the electromagnetic field as a source of curvature of spacetime in addition to the matter distribution. Apart from the ensuing generality, these features lead to forms of the virial theorem which are physically clearer, or at least more closely analogous to the Newtonian forms and the classical results of Chandrasekhar.

We shall confine our attention to the case of a diagonal metric tensor $g_{\mu\nu}$ and coordinates $x^\mu = (x^1, x^2, x^3, x^4)$ with $x^4 = ct$, c being the speed of light *in vacuo*. Greek index letters take the values 1, 2, 3, 4 for the spacetime coordinates, Roman ones the values 1, 2, 3 for the space coordinates, the signature of $g_{\mu\nu}$ is +2, a comma denotes partial differentiation, and a semicolon covariant differentiation. If we set $q = (-g)^{1/2}$, $h = \gamma^{1/2}$, $s = (-g_{44})^{1/2}$ where g and γ are the determinants of the spacetime metric

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tensor $g_{\mu\nu}$ and the spatial metric tensor γ_{ij} respectively, then $q = hs$. The Newtonian gravitational constant will be denoted by G, and we shall use electrostatic units for the electromagnetic field.

2. The equations of motion and Maxwell's equations

We consider a domain of curved spacetime with an energy distribution associated with a charged thermodynamic perfect fluid of null conductivity, and an electromagnetic field. The metric tensor $g_{\mu\nu}$ and the electromagnetic tensor $F_{\mu\nu}$ may be taken to arise from the fluid itself. The total energy tensor $T_\mu{}^\nu$ is $T_\mu{}^\nu = M_\mu{}^\nu + E_\mu{}^\nu$, where $M_\mu{}^\nu$ and $E_\mu{}^\nu$ are respectively the matter and electromagnetic energy tensors. The detailed form of $E_\mu{}^\nu$ is not directly required, but for $M_\mu{}^\nu$ we have

$$M_\mu{}^\nu = \mathring{n} m K V_\mu V^\nu + p\delta_\mu{}^\nu. \tag{2.1}$$

Here $\mathring{n}$ is the proper number density of particles of the fluid at x^α, m is the proper mass of a single particle, $K = (\mathring{\rho} + p/c^2)/\mathring{n}m$, $\mathring{\rho}$ and p being respectively the proper mass density and isotropic pressure of the fluid at x^α, $V^\mu = \mathrm{d}x^\mu/\mathrm{d}\tau$ is the contravariant form of the four-velocity of the fluid at x^α, $\mathrm{d}\tau = (-g_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta)^{1/2}/c$ being the element of proper time, and $\delta_\mu{}^\nu$ is the Kronecker delta. If n is the coordinate number density of particles at x^α, then $n = \mathring{n}s\,\mathrm{d}t/\mathrm{d}\tau$. Similarly, if σ and $\mathring{\sigma}$ are respectively the coordinate and proper charge densities at x^α, then $\sigma = \mathring{\sigma}s\,\mathrm{d}t/\mathrm{d}\tau$ (Møller 1972 p 415).

The particle four-current n^ν with norm $\mathring{n}c$, and the electric four-current J^ν with norm $\mathring{\sigma}c$ are given by $n^\nu = \mathring{n}V^\nu = n(v^i, c)/s$, and $J^\nu = \mathring{\sigma}V^\nu = \sigma(v^i, c)/s$, v^i being the contravariant form of the three-velocity of the fluid at x^α. The total energy tensor and the four-currents are covariantly conserved:

$$T_\mu{}^\nu{}_{;\nu} = 0, \tag{2.2}$$

$$J^\nu{}_{;\nu} = 0, \tag{2.3}$$

$$n^\nu{}_{;\nu} = 0. \tag{2.4}$$

Using equations (2.1), (2.4) and the results $E_\mu{}^\nu{}_{;\nu} = -F_{\mu\nu}J^\nu/c$, $V^\nu(KV_\mu)_{,\nu} = \mathrm{d}(KV_\mu)/\mathrm{d}\tau$, a more detailed form of equation (2.2) is

$$\mathring{n}m(\mathrm{d}/\mathrm{d}\tau)(KV_\mu) = \mathring{n}mK\Gamma^\beta{}_{\mu\nu}V^\nu V_\beta - p_{,\mu} + (1/c)F_{\mu\nu}J^\nu, \tag{2.5}$$

where $\Gamma^\beta{}_{\mu\nu}$ is the Christoffel symbol of the second kind. The spatial part of equation (2.5) may be written as

$$\mathring{n}m\frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}}{\mathrm{d}t}\left(K\frac{\mathrm{d}t}{\mathrm{d}\tau}v_i\right) = \mathring{n}mK\Gamma^\beta{}_{i\nu}V^\nu V_\beta - p_{,i} + \frac{1}{c}F_{i\nu}J^\nu, \tag{2.6}$$

where $\mathrm{d}/\mathrm{d}t = (\partial/\partial t) + (v^k\partial/\partial x^k) = (\mathrm{d}\tau/\mathrm{d}t)V^\beta\partial/\partial x^\beta$ represents time differentiation following the motion.

The usual four-tensor form of the electromagnetic field equations in curved spacetime is unsuitable here, and we shall express these equations in Maxwellian form in terms of the electromagnetic three-vectors $\boldsymbol{E}$, $\boldsymbol{D}$, $\boldsymbol{H}$, and $\boldsymbol{B}$. This is done in detail by Møller (1972 p 415), and so we shall only state the results in terms of the notation and convention used here. The electromagnetic tensor $F_{\mu\nu}$ is $F_{\mu\nu} = \Phi_{\nu,\mu} - \Phi_{\mu,\nu}$ where Φ_μ is the electromagnetic four-potential. The covariant forms of the three-vectors $\boldsymbol{E}$ and $\boldsymbol{H}$ are then given by $E_i = F_{i4} = \Phi_{4,i} - \Phi_{i,4}$, $H_i = se_{ikp}F^{kp}/2$, where $e_{ikp} = h\epsilon_{ikp}$ and $e^{ikp} =$

ϵ_{ikp}/h are the completely antisymmetric permutation tensors, ϵ_{ikp} being the Levi–Civita symbol. In time-orthogonal coordinates, $\gamma_{ij} = g_{ij}$ and the contravariant forms of $\boldsymbol{D}$ and $\boldsymbol{B}$ are given by $D^i = E^i/s$, $B^i = H^i/s$, where $E^i = \gamma^{ij}E_j$, $H^i = \gamma^{ij}H_j$. The covariant form of the vector product $(\boldsymbol{a}\times\boldsymbol{b})$ of two three-vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is defined by $(\boldsymbol{a}\times\boldsymbol{b})_i = e_{ikp}a^kb^p$, while the curl and divergence of $\boldsymbol{a}$ are defined by $\text{curl}\,\boldsymbol{a} = e^{ikp}a_{p,k}$ and $\text{div}\,\boldsymbol{a} = (ha^k)_{,k}/h$. With these, the equations of the electromagnetic field assume the form

$$\text{curl}\,\boldsymbol{E} = -h^{-1}(h\boldsymbol{B})_{,4}, \qquad \text{div}\,\boldsymbol{B} = 0, \tag{2.7}$$

$$\text{curl}\,\boldsymbol{H} - h^{-1}(h\boldsymbol{D})_{,4} = (4\pi/c)\sigma\boldsymbol{v}, \qquad \text{div}\,\boldsymbol{D} = 4\pi\sigma, \tag{2.8}$$

which are Maxwell's equations in curvilinear coordinates.

It is now possible to calculate the expression $qF_{i\nu}J^\nu/c$ which will be required in the sequel. Using the expressions for $F_{i\nu}$ and J^ν and the definition of the vector product, we obtain $qF_{i\nu}J^\nu/c = \sigma h[(\boldsymbol{v}\times\boldsymbol{B})_i/c + (\Phi_{4,i} - \Phi_{i,4})]$. Using the first of equations (2.8) and the definitions of the vector product and the curl operator, this is reduced to the desired form

$$\frac{1}{c}qF_{i\nu}J^\nu = h\left[\frac{1}{4\pi}\epsilon_{ikm}\epsilon_{knp}H_{p,n}B^m - \frac{1}{4\pi}\epsilon_{ikm}B^m(hD^k)_{,4} + \sigma(\Phi_{4,i} - \Phi_{i,4})\right]. \tag{2.9}$$

3. The virial theorem

We now multiply the equation of motion (2.6) by x^jq, and integrate over a spatial domain Ω. Using the expression (2.9) for $qF_{i\nu}J^\nu/c$, the relation between n and $\mathring{n}$, and bearing in mind that the element $\mathrm{d}\Omega$ of spatial volume is $\mathrm{d}\Omega = h\,\mathrm{d}^3x$ where $\mathrm{d}^3x = \mathrm{d}x^1\,\mathrm{d}x^2\,\mathrm{d}x^3$, we obtain

$$\int_\Omega x^jnm\frac{\mathrm{d}}{\mathrm{d}t}\left(K\frac{\mathrm{d}t}{\mathrm{d}\tau}v_i\right)\mathrm{d}\Omega = \mathfrak{W}_i^j + \mathfrak{Z}_i^j + \mathfrak{Q}_i^j - \int_\Omega x^jsp_{,i}\,\mathrm{d}\Omega + \frac{1}{4\pi}\int_\Omega x^j\epsilon_{ikm}\epsilon_{knp}H_{p,n}B^m\,\mathrm{d}\Omega, \tag{3.1}$$

where

$$\mathfrak{W}_i^j = \int_\Omega x^jnm\frac{\mathrm{d}\tau}{\mathrm{d}t}K\Gamma^\beta{}_{i\nu}V^\nu V_\beta\,\mathrm{d}\Omega, \tag{3.2}$$

$$\mathfrak{Z}_i^j = \int_\Omega x^j\sigma\Phi_{4,i}\,\mathrm{d}\Omega, \tag{3.3}$$

$$\mathfrak{Q}_i^j = -\int_\sigma x^j\left\{\frac{1}{4\pi}\epsilon_{ikm}B^m(hD^k)_{,4} + \sigma\Phi_{i,4}\right\}\mathrm{d}\Omega. \tag{3.4}$$

The element of proper spatial volume is $\mathrm{d}\mathring{\Omega} = s(\mathrm{d}t/\mathrm{d}\tau)\,\mathrm{d}\Omega$, and so $n\,\mathrm{d}\Omega = \mathring{n}\,\mathrm{d}\mathring{\Omega}$. Using the results $\mathrm{d}\mathring{\Omega}/\mathrm{d}\tau = V^\nu{}_{;\nu}\,\mathrm{d}\mathring{\Omega}$ and $\mathrm{d}\mathring{n}/\mathrm{d}\tau = V^\nu\mathring{n}_{,\nu}$ (Misner *et al* 1973 p 559), we obtain $\mathrm{d}(\mathring{n}\,\mathrm{d}\mathring{\Omega})/\mathrm{d}\tau = n^\nu{}_{;\nu}\,\mathrm{d}\mathring{\Omega} = 0$ by equation (2.4). Since $n\,\mathrm{d}\Omega = \mathring{n}\,\mathrm{d}\mathring{\Omega}$ and $\mathrm{d}/\mathrm{d}\tau = (\mathrm{d}t/\mathrm{d}\tau)\,\mathrm{d}/\mathrm{d}t$, this implies that $\mathrm{d}(n\,\mathrm{d}\Omega)/\mathrm{d}t = 0$, and so equation (3.1) may be written as

$$\frac{\mathrm{d}\mathfrak{L}_i^j}{\mathrm{d}t} = \mathfrak{W}_i^j + 2\mathfrak{T}_i^j + \mathfrak{Z}_i^j + \mathfrak{Q}_i^j - \int_\Omega x^jsp_{,i}\,\mathrm{d}\Omega + \frac{1}{4\pi}\int_\Omega x^j\epsilon_{ikm}\epsilon_{knp}H_{p,n}B^m\,\mathrm{d}\Omega, \tag{3.5}$$

where

$$\mathfrak{L}_i^j = \int_\Omega x^j nmK\frac{dt}{d\tau}v_i \, d\Omega, \tag{3.6}$$

$$\mathfrak{T}_i^j = \frac{1}{2}\int_\Omega nmK\frac{dt}{d\tau}v_i v^j \, d\Omega. \tag{3.7}$$

The last two volume integrals in equation (3.5) may now be transformed into sums of volume and surface integrals. Using the result $q_{,i} = q\Gamma^\beta{}_{i\beta}$ (Møller 1972 p 339), we find

$$\int_\Omega x^j sp_{,i} \, d\Omega = \int_{\partial\Omega} x^j sp \, d\Sigma_i - \mathfrak{O}_i^j - \delta_i^j(\Gamma-1)\mathfrak{U} \tag{3.8}$$

where

$$\mathfrak{O}_i^j = \int_\Omega x^j sp\Gamma^\beta{}_{i\beta} \, d\Omega, \tag{3.9}$$

$$\mathfrak{U} = \frac{1}{\Gamma-1}\int_\Omega sp \, d\Omega. \tag{3.10}$$

Here Γ is used to denote the adiabatic index in the context of the Newtonian reduction of $\delta_i^j(\Gamma-1)\mathfrak{U}$, to be carried out later, $\partial\Omega$ is the boundary of Ω and $d\Sigma_i$ is the component of the vector element of area along x^i. The transformation of the last volume integral in equation (3.5), is more complicated. After some calculations involving the Levi–Civita symbol, the second of Maxwell's equations (2.7) and the result $h_{,i} = h\Gamma^p{}_{ip}$, we obtain

$$\begin{aligned}\int_\Omega x^j \epsilon_{ikm}\epsilon_{knp}H_{p,n}B^m \, d\Omega &= \int_{\partial\Omega} x^j(H_iB^k \, d\Sigma_k - H_kB^k \, d\Sigma_i) + \delta_i^j\int_\Omega H_kB^k \, d\Omega \\ &\quad - \int_\Omega H_iB^j \, d\Omega + \int_\Omega x^jH_kB^k\Gamma^p{}_{ip} \, d\Omega + \int_\Omega x^jH_kB^k{}_{,i} \, d\Omega.\end{aligned} \tag{3.11}$$

We further note that $x^j(H_{i,k}B^k - H_{k,i}B^k - \epsilon_{ikm}\epsilon_{knp}H_{p,n}B^m)$ is null for all i and j, and so its volume integral is null. This leads to a second expression for the last volume integral in equation (3.5):

$$\int_\Omega x^j\epsilon_{ikm}\epsilon_{knp}H_{p,n}B^m \, d\Omega = \int_{\partial\Omega} x^jH_iB^k \, d\Sigma_k - \int_\Omega H_iB^j \, d\Omega - \int_\Omega x^jH_{k,i}B^k \, d\Omega. \tag{3.12}$$

Combining expressions (3.11) and (3.12), we finally obtain

$$\frac{1}{4\pi}\int_\Omega x^j\epsilon_{ikm}\epsilon_{knp}H_{p,n}B^m \, d\Omega = \frac{1}{8\pi}\int_{\partial\Omega} x^j(2H_iB^k \, d\Sigma_k - H_kB^k \, d\Sigma_i) + \delta_i^j\mathfrak{M} - 2\mathfrak{M}_i^j + \mathfrak{N}_i \tag{3.13}$$

where

$$\mathfrak{M}_i^j = \frac{1}{8\pi}\int_\Omega H_iB^j \, d\Omega, \tag{3.14}$$

$$\mathfrak{N}_i^j = \frac{1}{8\pi}\int_\Omega x^j(H_kB^k\Gamma^p{}_{ip} + H_kB^k{}_{,i} - H_{k,i}B^k) \, d\Omega, \tag{3.15}$$

and $\mathfrak{M}$ is the contracted form of $\mathfrak{M}_i^j$. Using equations (3.8) and (3.13), equation (3.5) becomes

$$\frac{\mathrm{d}\mathfrak{L}_i^j}{\mathrm{d}t}=\mathfrak{W}_i^j+2\mathfrak{T}_i^j+\mathfrak{Z}_i^j+\mathfrak{Q}_i^j-\int_{\partial\Omega} x^j sp\ \mathrm{d}\Sigma_i+\mathfrak{O}_i^j+\delta_i^j(\Gamma-1)\mathfrak{U}$$
$$+\frac{1}{8\pi}\int_{\partial\Omega} x^j(2H_iB^k\ \mathrm{d}\Sigma_k-H_kB^k\ \mathrm{d}\Sigma_i)+\delta_i^j\mathfrak{M}-2\mathfrak{M}_i^j+\mathfrak{N}_i^j. \tag{3.16}$$

The contracted form of this equation is

$$\frac{\mathrm{d}\mathfrak{L}}{\mathrm{d}t}=\mathfrak{W}+2\mathfrak{T}+\mathfrak{Z}+\mathfrak{Q}-\int_{\partial\Omega} x^i sp\ \mathrm{d}\Sigma_i+\mathfrak{O}+3(\Gamma-1)\mathfrak{U}$$
$$+\frac{1}{8\pi}\int_{\partial\Omega} x^i(2H_iB^k\ \mathrm{d}\Sigma_k-H_kB^k\ \mathrm{d}\Sigma_i)+\mathfrak{M}+\mathfrak{N}. \tag{3.17}$$

We note that the terms appearing in these equations are not tensors, and that the Christoffel symbol of the second kind appearing in the expression for $\mathfrak{N}_i^j$ is with respect to the spatial metric tensor γ_{ij}.

Physical considerations imply that if the boundary is placed at infinity, the surface integrals in equations (3.16) and (3.17) vanish, and so

$$\frac{\mathrm{d}\mathfrak{L}_i^j}{\mathrm{d}t}=\mathfrak{W}_i^j+2\mathfrak{T}_i^j+\mathfrak{Z}_i^j+\mathfrak{Q}_i^j+\mathfrak{O}_i^j+\delta_i^j(\Gamma-1)\mathfrak{U}+\delta_i^j\mathfrak{M}-2\mathfrak{M}_i^j+\mathfrak{N}_i^j, \tag{3.18}$$

$$\frac{\mathrm{d}\mathfrak{L}}{\mathrm{d}t}=\mathfrak{W}+2\mathfrak{T}+\mathfrak{Z}+\mathfrak{Q}+\mathfrak{O}+3(\Gamma-1)\mathfrak{U}+\mathfrak{M}+\mathfrak{N}. \tag{3.19}$$

Equation (3.18) and its contracted version (3.19) represent exact forms of the virial theorem in curved spacetime, which are sufficiently general to be applied to a thermodynamic perfect fluid or dust cloud and to a charged thermodynamic perfect fluid or dust cloud in an electromagnetic field. In the absence of an electric field, the terms $\mathfrak{Z}_i^j$ and $\mathfrak{Q}_i^j$ and their contracted forms in equations (3.16)–(3.19) vanish, and we have the case of hydromagnetics in curved spacetime. In the absence of a magnetic field, the terms $\mathfrak{Q}_i^j$, $\mathfrak{M}_i^j$, and $\mathfrak{N}_i^j$ and their contracted forms in equations (3.16)–(3.19) as well as the surface integrals involving the components of the magnetic three-vectors in equations (3.16)–(3.17) vanish, and we have the case of a charged thermodynamic perfect fluid in an electric field in curved spacetime. In the absence of both electric and magnetic fields, the terms $\mathfrak{Z}_i^j$, $\mathfrak{Q}_i^j$, $\mathfrak{M}_i^j$ and $\mathfrak{N}_i^j$ and their contracted forms in equations (3.16)–(3.19) as well as the surface integrals involving the components of the magnetic three-vectors in equations (3.16)–(3.17) vanish, and we have the case of a thermodynamic perfect fluid in curved spacetime.

An alternative version of equations (3.16)–(3.19) which perhaps gives a clearer physical picture may be obtained by re-expressing the terms $\mathfrak{Z}_i^j$ and $\mathfrak{Q}_i^j$. We first note that $E_{i,k}=E_{k,i}+F_{ik,4}$, and on using the second of Maxwell's equations (2.8), $\sigma(\Phi_{4,i}-\Phi_{i,4})=E_i(hD^k)_{,k}/4\pi h$. We may thus obtain

$$\mathfrak{Z}_i^j+\mathfrak{Q}_i^j=\frac{1}{4\pi}\bigg(\int_{\partial\Omega} x^jE_iD^k\ \mathrm{d}\Sigma_k-\int_\Omega E_iD^j\ \mathrm{d}\Omega-\int_\Omega x^jE_{k,i}D^k\ \mathrm{d}\Omega-\int_\Omega x^j(\boldsymbol{D}\times\boldsymbol{B})_{i,4}\ \mathrm{d}\Omega$$
$$-\int_\Omega x^j(\boldsymbol{D}\times\boldsymbol{B})_ih^{-1}h_{,4}\ \mathrm{d}\Omega\bigg), \tag{3.20}$$

where we have used the relations $F_{ik,4}D^k = \epsilon_{ikm}(hB^m)_{,4}D^k$, and $\epsilon_{ikm}((hD^k)_{,4}B^m + D^k(hB^m)_{,4}) = (\boldsymbol{D}\times\boldsymbol{B})_{i,4} + (\boldsymbol{D}\times\boldsymbol{B})_i h^{-1}h_{,4}$ with $h_{,4} = \frac{1}{2}h\gamma^{pk}\gamma_{pk,4}$. By substituting $x^j hE_{k,i}D^k = (x^j hE_kD^k)_{,i} - \delta_i^j hE_kD^k - x^j\Gamma^p{}_{ip}hE_kD^k - x^j hE_kD^k{}_{,i}$ with $\Gamma^p{}_{ip}h = h_{,i}$ into equation (3.20), we also have

$$\mathfrak{Z}_i{}^j + \mathfrak{Q}_i{}^j = \frac{1}{4\pi}\bigg(\int_{\partial\Omega} x^j E_i D^k \,\mathrm{d}\Sigma_k - \int_\Omega E_i D^j \,\mathrm{d}\Omega - \int_{\partial\Omega} x^j E_k D^k \,\mathrm{d}\Sigma_i + \delta_i^j \int_\Omega E_k D^k \,\mathrm{d}\Omega + \int_\Omega x^j E_k D^k \Gamma^p{}_{pi}\,\mathrm{d}\Omega + \int_\Omega x^j E_k D^k{}_{,i}\,\mathrm{d}\Omega - \int_\Omega x^j(\boldsymbol{D}\times\boldsymbol{B})_{i,4}\,\mathrm{d}\Omega - \int_\Omega x^j(\boldsymbol{D}\times\boldsymbol{B})_i h^{-1}h_{,4}\,\mathrm{d}\Omega\bigg). \quad (3.21)$$

Combining equations (3.20) and (3.21) we finally obtain

$$\mathfrak{Z}_i{}^j + \mathfrak{Q}_i{}^j = \frac{1}{8\pi}\int_{\partial\Omega} x^j(2E_iD^k\,\mathrm{d}\Sigma_k - E_kD^k\,\mathrm{d}\Sigma_i) + \delta_i^j\mathfrak{E} - 2\mathfrak{E}_i{}^j + \mathfrak{K}_i{}^j - \frac{1}{4\pi}\int_\Omega x^j(\boldsymbol{D}\times\boldsymbol{B})_{i,4}\,\mathrm{d}\Omega - \frac{1}{4\pi}\int_\Omega x^j(\boldsymbol{D}\times\boldsymbol{B})_i h^{-1}h_{,4}\,\mathrm{d}\Omega, \quad (3.22)$$

where

$$\mathfrak{E}_i{}^j = \frac{1}{8\pi}\int_\Omega E_iD^j\,\mathrm{d}\Omega, \quad (3.23)$$

$$\mathfrak{K}_i{}^j = \frac{1}{8\pi}\int_\Omega x^j(E_kD^k\Gamma^p{}_{pi} + E_kD^k{}_{,i} - E_{k,i}D^k)\,\mathrm{d}\Omega, \quad (3.24)$$

and $\mathfrak{E}$ is the contracted form of $\mathfrak{E}_i{}^j$. Thus equation (3.16) may also be written as

$$\frac{\mathrm{d}\mathfrak{L}_i{}^j}{\mathrm{d}t} = \mathfrak{W}_i{}^j + 2\mathfrak{T}_i{}^j - \int_{\partial\Omega} x^j sp\,\mathrm{d}\Sigma_i + \mathfrak{D}_i{}^j + \delta_i^j(\Gamma - 1)\mathfrak{U} + \frac{1}{8\pi}\int_{\partial\Omega} x^j[2(H_iB^k + E_iD^k)\,\mathrm{d}\Sigma_k - (H_kB^k + E_kD^k)\,\mathrm{d}\Sigma_i] + \delta_i^j(\mathfrak{M} + \mathfrak{E}) - 2(\mathfrak{M}_i{}^j + \mathfrak{E}_i{}^j) - \frac{1}{4\pi}\int_\Omega x^j(\boldsymbol{D}\times\boldsymbol{B})_{i,4}\,\mathrm{d}\Omega - \frac{1}{4\pi}\int_\Omega x^j(\boldsymbol{D}\times\boldsymbol{B})_i h^{-1}h_{,4}\,\mathrm{d}\Omega + \mathfrak{N}_i{}^j + \mathfrak{K}_i{}^j. \quad (3.25)$$

This may be contracted by taking indexes i and j to be equal to give a form analogous to equation (3.17). As before the surface integrals in (3.25) and its contracted form will vanish if the boundary is placed at infinity.

Equations (3.16) and (3.25) are both useful; equation (3.25) is easier to interpret physically. We may identify the quantity $(H_kB^k + E_kD^k)/8\pi$ in the volume integral expression for $\mathfrak{M} + \mathfrak{E}$ as the electromagnetic energy density. The quantities $(\boldsymbol{D}\times\boldsymbol{B})_i/4\pi c$ and $(\boldsymbol{D}\times\boldsymbol{B})_{i,4}/4\pi$ occurring in two other volume integrals are respectively the electromagnetic momentum density and its time derivative. The volume and surface integrals involving the components of the magnetic three-vectors, are similar to the corresponding terms of Chandrasekhar's hydromagnetics, (Chandrasekhar 1960, 1961). This similarity is purely formal, however, as in the present case the effects of gravitation, namely the distortion of spacetime, are implicitly included in all the terms of all the equations. We may show that the terms $\mathfrak{N}_i^{\ j}$ and $\mathfrak{K}_i^{\ j}$ arise from the derivatives of the components of the spatial metric tensor. Consequently, these terms, as well as all the terms containing Christoffel symbols, vanish only in Cartesian coordinates in flat spacetime, giving rise to special relativistic forms of the equations.

The physical significance of the remaining terms is best seen if we attempt a Newtonian reduction of the expressions. To find the Newtonian limit we expand the terms of these equations up to the first inverse power of c. This implies that we take $\mathring{\rho} = \mathring{n}m$ and $K = 1$; we further take $g_{ij} = \delta_i^{\ j}$, $g_{44} = -1$, $h = 1$ and $dt/d\tau = 1$ for all the terms, except that in calculating $\Gamma^{\beta}{}_{i\nu} V^{\nu} V_{\beta}$ in the expression for $\mathfrak{W}_i^{\ j}$, we take $g_{ij} = (1 + 2\mathfrak{B}/c^2)\delta_i^{\ j}$, $g_{44} = -(1 - 2\mathfrak{B}/c^2)$ where $\mathfrak{B}$ is the Newtonian gravitational potential. With these, referring to equation (3.25), the terms $h_{,4}$, $\mathfrak{Q}_i^{\ j}$, $\mathfrak{N}_i^{\ j}$ and $\mathfrak{K}_i^{\ j}$ reduce to zero, $d\Omega$ and $d\Sigma_i$ reduce to their flat space expressions in Cartesian coordinates d^3x and dS_i respectively, and equation (3.25) reduces to

$$\frac{d}{dt}\int_{\Omega} x^j \mathring{\rho} v^i \, d^3x = \int_{\Omega} x^j \mathring{\rho} \mathfrak{B}_{,i} \, d^3x + \int_{\Omega} \mathring{\rho} v^i v^j \, d^3x - \int_{\Omega} x^j p \, dS_i + \delta_i^{\ j} \int_{\Omega} p \, d^3x$$
$$+ \frac{1}{8\pi} \int_{\partial\Omega} x^j [2(H^i B^k + E^i D^k) \, dS_k - (H^k B^k + E^k D^k) \, dS_i]$$
$$+ \delta_i^{\ j}(\mathfrak{M} + \mathfrak{E}) - 2(\mathfrak{M}_i^{\ j} + \mathfrak{E}_i^{\ j}) - \frac{1}{4\pi} \int_{\Omega} x^j (\boldsymbol{D} \times \boldsymbol{B})_{i,4} \, d^3x, \qquad (3.26)$$

where the terms appear in the same order as in equation (3.25), and where we have used the fact that for any three-vector $\boldsymbol{a}$, $a_i = a^i$. All the terms in equation (3.26) are now tensors with obvious physical meanings. Thus the terms $\mathfrak{W}_i^{\ j}$ and $\mathfrak{T}_i^{\ j}$ have reduced to the volume integrals of $x^j \mathring{\rho} \mathfrak{B}_{,i}$ and $\mathring{\rho} v^i v^j/2$, which are the Newtonian gravitational potential energy tensor and Newtonian kinetic energy tensor respectively. It may be shown that the left-hand side of equation (3.26) may be written as the second time derivative of the inertia tensor, which is the volume integral of $\mathring{\rho} x^i x^j$. Thus the time derivative of $\mathfrak{L}_i^{\ j}$ has reduced to the second time derivative of the inertia tensor in the Newtonian approximation. Similar reductions hold for the remaining equations. These reduced equations describe a Newtonian fluid in an electromagnetic field and Newtonian gravitational field. If the electric field vanishes they reduce exactly to Chandrasekhar's equations in hydromagnetics.

4. Applications

As a first application of the exact virial theorem in curved spacetime (equation (3.18)) we consider a stationary self-gravitating spherically symmetric charged dust cloud in static equilibrium. In the absence of external fields $g_{\mu\nu}$ and $F_{\mu\nu}$ arise solely from the dust cloud. If the line element in the domain of spacetime occupied by the dust cloud is $ds^2 = (dr^2 + r^2 \, d\theta^2 + r^2 \sin^2\theta \, d\phi^2)/f^2 - c^2 f^2 \, dt^2$, where f is a function of r only, (r, θ, ϕ) being spherical polar coordinates, we have: for a dust cloud $K = 1$, and so $\mathring{\rho} = \mathring{n}m$; $p = 0$, and so $\mathfrak{Q}_i^{\ j} = \mathfrak{U} = 0$. Since $\boldsymbol{H} = \boldsymbol{0}$, in time-orthogonal coordinates $\boldsymbol{B} = \boldsymbol{0}$. This, together with the result $\Phi_{i,4} = 0$ implies that $\mathfrak{Q}_i^{\ j} = \mathfrak{M}_i^{\ j} = \mathfrak{M} = \mathfrak{N}_i^{\ j} = 0$. Since the dust cloud is in static equilibrium, $d\mathfrak{L}_i^{\ j}/dt = \mathfrak{T}_i^{\ j} = 0$, and so the 11 component of equation

(3.18) gives $\mathfrak{W}_1{}^1+\mathfrak{Z}_1{}^1=0$. With the above line element, $s=f$, and so $n=\mathring{n}f\,\mathrm{d}t/\mathrm{d}\tau$, $\sigma=\mathring{\sigma}f\,\mathrm{d}t/\mathrm{d}\tau$. Thus using the expressions (3.2) and (3.3) for $\mathfrak{W}_i{}^j$ and $\mathfrak{Z}_i{}^j$ respectively, we find that the equation $\mathfrak{W}_1{}^1+\mathfrak{Z}_1{}^1=0$ expresses the fact that the integral of $rf(\mathring{\rho}\Gamma^{\beta}_{1\nu}V^{\nu}V_{\beta}+\mathring{\sigma}\Phi_{4,i}\,\mathrm{d}t/\mathrm{d}\tau)$ over Ω vanishes, where the volume element $\mathrm{d}\Omega$ is given by $\mathrm{d}\Omega=h\,\mathrm{d}^3x=r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi/f^3$. Since $\mathrm{d}r/\mathrm{d}t$, $\mathrm{d}\theta/\mathrm{d}t$, and $\mathrm{d}\phi/\mathrm{d}t$ vanish in static equilibrium, calculation shows that

$$4\pi\int f^{-3}r^3(\mathring{\rho}c^2f_{,1}+\mathring{\sigma}\Phi_{4,1})\,\mathrm{d}r=0, \tag{4.1}$$

where we have partially integrated over θ and ϕ.

Equation (4.1) represents an equilibrium condition for the charged matter considered, in terms of total integrated effects. In particular, condition (4.1) will be satisfied if

$$\mathring{\sigma}=\pm\mathring{\rho}, \qquad c^2\frac{\mathrm{d}f}{\mathrm{d}r}=\mp\frac{\mathrm{d}\Phi_4}{\mathrm{d}r}. \tag{4.2}$$

It may actually be shown that we should set $\mathring{\sigma}=\pm\mathring{\rho}G^{1/2}$ and $(c^2/G^{1/2})\,\mathrm{d}f/\mathrm{d}r=\mp\mathrm{d}\Phi_4/\mathrm{d}r$. The condition (4.2), which is a special case of equation (4.1), was also obtained by Bonnor using different considerations and the field equations (Bonnor 1965). These results indicate the role of charge in the equilibrium of mass distributions and show the existence of equilibrium configurations. They also indicate the possible relevance of charge to gravitational collapse.

As a second application, we consider the equilibrium of charged elastic matter. The line element will be the same as in the first application, but the total energy tensor will contain the additional term $S_\mu{}^\nu$ which is associated with the elastic properties of the matter. The detailed form of $S_\mu{}^\nu$ is given by Rayner (1963). It follows that the equilibrium condition is of the form (4.1), but with the additional term $S_1{}^\nu{}_{;\nu}r^3/f^2$ in the integrand. Since the only non-vanishing components of $S_\mu{}^\nu$ are $S_1{}^1=S_2{}^2=S_3{}^3=-(1+f^2)(2\mu+3\nu)/2$, where $\mu(r)$ and $\nu(r)$ are the Rayner scalars associated with the elast c matter (Nduka 1975), we have $S_1{}^\nu{}_{;\nu}=-[(f+f^3)(2\mu+3\nu)]_{,1}/2f$. With these, the equilibrium condition is

$$\int f^{-3}r^3\left\{\mathring{\rho}c^2f_{,1}+\mathring{\sigma}\Phi_{4,1}-\frac{1}{2}[(f+f^3)(2\mu+3\nu)]_{,1}\right\}\mathrm{d}r=0. \tag{4.3}$$

It follows that a special case of equation (4.3) is

$$\mathring{\sigma}=\pm\mathring{\rho}, \qquad c^2\frac{\mathrm{d}f}{\mathrm{d}r}=\mp\frac{\mathrm{d}\Phi_4}{\mathrm{d}r}, \qquad \mu=-\frac{3}{2}\nu. \tag{4.4}$$

The particular case of equation (4.4) confirms a result of Nduka, who used Bonnor's method for the equilibrium of charged elastic matter (Nduka 1975).

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